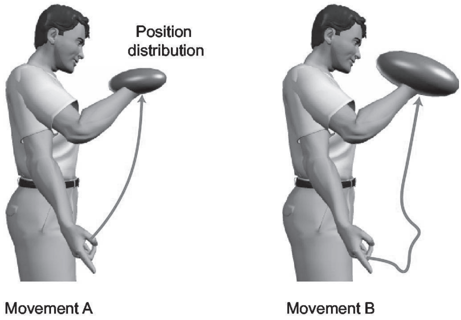
Optimal Control

Maneesh Sahani

November 1, 2019

The problem of movement



How do we move to achieve a goal?

- complicated by flexibility
- final "goal" posture depends on start point
- rapid accommodation to feedback and perturbation

Optimization principles

Of all the different ways to achieve a goal, we often seem to choose the *smoothest*.

- Flash & Hogan (1985) – minimum jerk (third derivative) of effector such as hand (or gaze)

$$\min \int_{t_0}^{t_f} dt \, \left(\frac{d^3x}{dt^3}\right)^2 + \left(\frac{d^3y}{dt^3}\right)^2$$

- Uno, Kawato & Suzuki (1989) – minimum rate of change of joint torque.
  - joints, not effector
  - muscle outputs not kinematics
  - solution will depend on musculoskelatal transformations
  - predicts path curvature
- Nakano et al (1999) elaborated to minimum *commanded* torque change – includes a model of muscles

But *why* should movements be smooth in this way? And what does this have to do with achieving a goal?

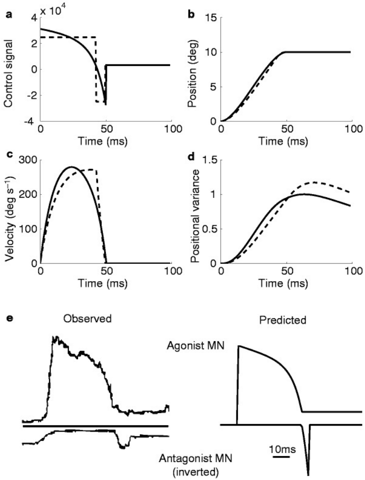
Optimality of outcome

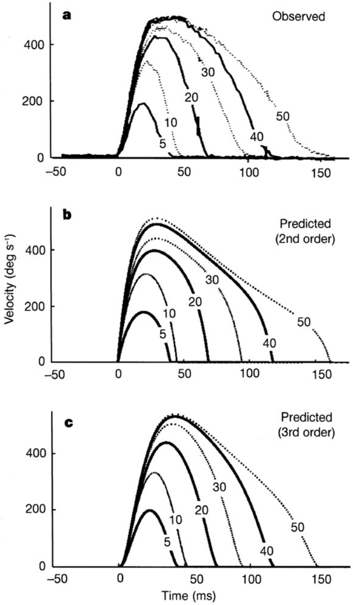
Harris & Wolpert (1998) suggested that planning to achieve outcome with minimum error in the face of *signal dependent noise* leads to smoothness.

$$
\begin{aligned}
&\mathbf{x}_{t+1} = A\mathbf{x}_t + B(u_t + w_t) \\
&w_t \sim \mathcal{N}\left(0, k u_t^2\right) \\
&\Rightarrow \mathbf{x}_T = \mathcal{N}\left(A^T\mathbf{x}_0 + \sum_{\tau=0}^{T-1} A^{T-1-\tau}Bu_\tau, k\sum_{\tau=0}^{T-1} A^{T-1-\tau}B(A^{T-1-\tau}B)^{\mathsf{T}}u_\tau^2\right)
\end{aligned}
$$

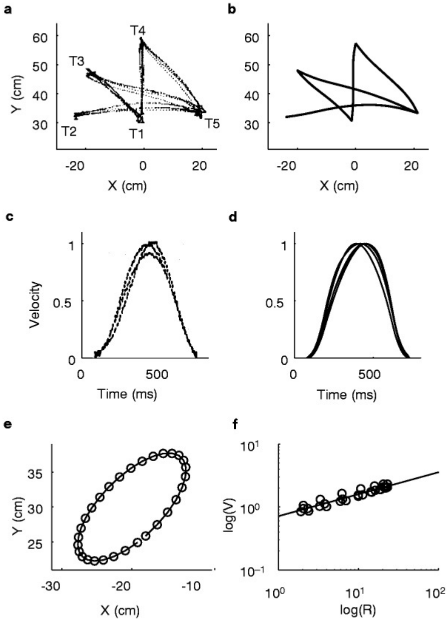
Choose control sequence to minimise expected error in endpoint at T; examine trajectory.

Optimal path control





Optimal path control



Feedback Control

Todorov & Jordan (2002) incorporate signal-dependent noise into a feedback control model.

- noise suppressed in relevant dimensions of outcome
- noise (and initial conditions) in *irrelevant dimensions* uncontrolled, and may increase
- predicts both trajectories *and* systematic and noisy variability

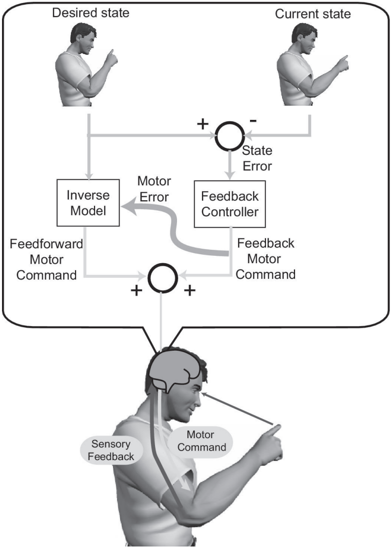
Optimal stochastic control:

- cost-to-go (value) based planning: Hamilton-Bellman-Jacobi equation

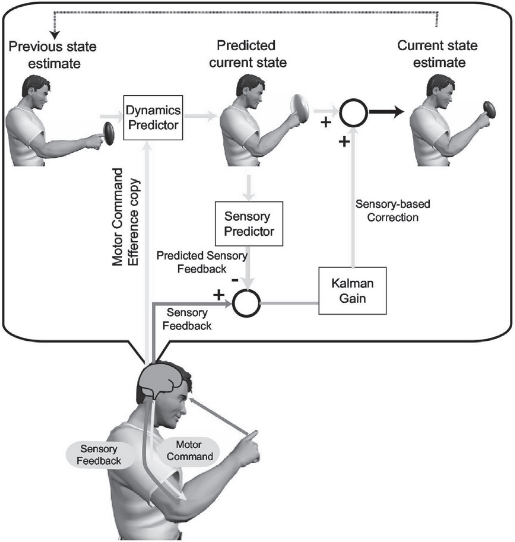
  $$S(x_t) = C(u_t) + \mathbb{E}_{p(x_{t+1}|x_t,u_t)}\left[S(x_{t+1})\right]$$

  minimising over u_t defines optimal control policy
- state-estimation: Kalman filtering

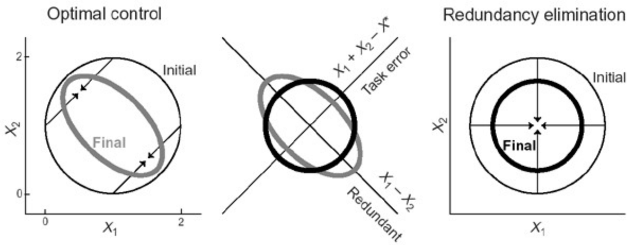
Controllers

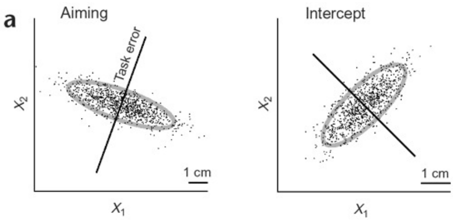


State estimation

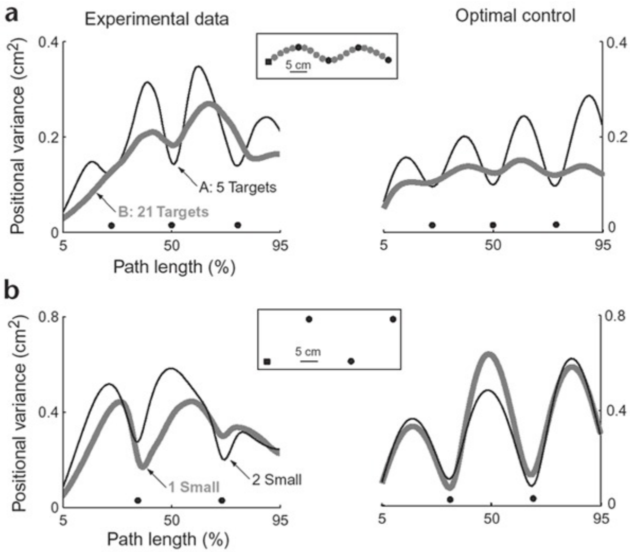


Shaped variance





Path variance



Path variance